Monatsh Math (2009) 158:97–102
DOI 10.1007/s00605-008-0090-y

On finiteness of chains of intermediate rings

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Received: 2 March 2008 / Accepted: 2 December 2008 / Published online: 8 January 2009

Abstract An extension of integral domains $R \subseteq S$ is said to have the "finite length of intermediate chains of domains" property (for short FICP) if each chain of intermediate rings between R and S is finite. The main purpose of this paper is to characterize when $R \subseteq S$ has FICP in case R^* (the integral closure of R in S) is a finite dimensional semilocal domain. This generalizes a theorem due to Gilmer, in which S is the quotient field of R. Examples illustrating the sharpness and the limits of our results are settled.

Keywords Ring extension · Intermediate chains of domains · Normal pair · Valuation domain

Mathematics Subject Classification (2000) Primary 13B02 · 13B22 · 13F05 · 13G05

1 Introduction

All rings considered below are integral, commutative with identity; and all subrings are unital. If R is an integral domain, let $qf(R)$ denote the quotient field of R and R' the integral closure of R. If $R \subseteq S$ is a ring extension, denote by R^* the integral closure of R in S and by $[R, S]$ the set of all rings T such that $R \subseteq T \subseteq S$.

A frequent theme in the study of intermediate rings between R and S is the finiteness of the length of chains of intermediate rings between R and S (cf. [1,5,6]). This

Communicated by D. Segal.

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property is labeled as (FICP): "finite length of intermediate chains of domains" property. Our work is motivated by two papers. The first is [6] in which Jaballah studied the finiteness of length of chains in $[R, S]$ when (R, S) is a normal pair. Our second motivation is [5], in which Gilmer both developed several techniques useful for studying FICP in $[R, qf(R)]$ and derived characterizations of such domains. The main result in this paper generalizes the work of Gilmer [5] who showed, among other things, that if R is an integral domain with quotient field K, then $R \subseteq K$ has FICP iff $R \subseteq R'$ and $R' \subseteq K$ have FICP.

As the title of this paper suggests, our goal is to obtain a necessary and sufficient condition for the length of chains of intermediate rings to be finite. Precisely, Theorem 2.4 states that if $R \subseteq S$ is a ring extension such that R^* is a semilocal domain with finite Krull dimension, then $R \subseteq S$ has FICP iff each of the extensions $R \subseteq R^*$ and $R^* \subseteq S$ has FICP. This generalizes Gilmer's study, in which $S = qf(R)$.

In addition to the above notations and conventions, denote by Spec (R) (resp., $\mathrm{Max}(R)$) the set of prime (resp., maximal) ideals of R. $P \subseteq Q$ are two prime ideals of R, let $[P, Q]$ denote the set of all prime ideals Q' of R such that $P \subseteq Q' \subseteq Q$. As usual, $\subset$ denotes proper inclusion. Any unexplained terminology is standard as in [7].

2 Main results

As an initial step toward understanding what extensions $R \subseteq S$ have FICP, we will show that this class is closed under localization and factor domains. The simple proof of this result is included for the sake of completeness.

Proposition 2.1 *Let $R \subseteq S$ be a ring extension. Assume that $R \subseteq S$ has FICP, then the following hold*:

(i) *For each multiplicative closed subset N of R, $N^{-1}R \subseteq N^{-1}S$ has FICP.*
(ii) *For each $Q \in Spec(S)$, setting $P = Q \cap R$ then $R/P \subseteq S/Q$ has FICP.*

Proof (i) and (ii). Let N be a multiplicative subset of R. For a prime ideal Q of S with contraction P in R , let $\varphi : S \to S/Q$ the canonical surjection. If $T_1 \in [N^{-1}R, N^{-1}S]$ (resp., $[R/P, S/Q]$), then $T_1 \cap S$ (resp., $\varphi^{-1}(T_1)) \in [R, S]$ and a straightforward calculation shows that $T_1 = N^{-1}(T_1 \cap S)$ (resp., $T_1 = \varphi^{-1}(T_1)/Q$). Hence, the assignment $T_1 \mapsto T_1 \cap S$ (resp., $T_1 \mapsto \varphi^{-1}(T_1)$) gives an increasing injection from $[N^{-1}R, N^{-1}S]$ (resp., $[R/P, S/Q]$) to $[R, S]$. As the latter extension has FICP, so is the former. □

In view of Proposition 2.1(i), it is natural to ask whether "locally FICP" implies FICP. The following example answers the question in the negative, while giving a positive answer if the basic ring is semilocal domain (see Proposition 2.3).

We recall that if $R \subset S$ is a pair of rings and P is a prime ideal of R, then we mean by S_P the ring S_N where $N = R - P$.

Example 2.2 Let R be a one dimensional Prüfer domain with infinitely many maximal ideals and take $S = qf(R)$. Notice that, for each maximal ideal M of R each chain of rings between R_M and S_M has length at most 2. On the other hand, let $\{M_i, i \geq 0\}$ be a subset of maximal ideals of R and take $T_i = \cap_{0 \leq k \leq i} R_{M_k}$. Then $T_0 \supset T_1 \supset \ldots$ is a strictly descending chain of intermediate rings between R and S.

Proposition 2.3 *Let R be an integral domain such that $Max(R)$ is finite. Then $R \subseteq S$ has FICP if and only if $R_M \subseteq S_M$ has FICP for $M \in Max(R)$.*

Proof If $R \subseteq S$ satisfies FICP, then so is $R_M \subseteq S_M$ for each maximal ideal M of R [Proposition 2.1(i)]. Conversely, suppose $\mathrm{Max}(R) = \{M_1, \ldots, M_n\}$ and assume that $R_{M_i} \subseteq S_{M_i}$ has FICP for each i. Because $T = \cap_{1 \leq i \leq n} T_{M_i}$ for each $T \in [R, S]$, it follows that $T = T_1 \cap \ldots \cap T_n$ where $T_i \in [R_{M_i}, S_{M_i}]$. Hence $[R, S]$ has FICP, as was to be proved. □

Before presenting our main result, we recall that if $R \subseteq S$ is a ring extension then (R, S) is said to be a *residually algebraic pair* if for any ring T in $[R, S]$ and any prime Q of T, T/Q is algebraic over $R/(Q \cap R)$ [2, Definitions 1.1 and 2.1]. The pair (R, S) is said to be *normal* if each $T \in [R, S]$ is integrally closed in S [4]. Moreover it was shown in [2, Theorem 2.10] that (R, S) is a normal pair iff it is residually algebraic and R is integrally closed in S.

The proof of Theorem 2.4 depends ultimately on normal pairs techniques. So it is convenient to recall some useful facts about these pairs [2, Theorem 2.5(v), Lemma 3.1(iii) and Lemma 2.9(i)]. Let (R, S) be a normal pair. Set $\mathrm{Max}(R) = \{M_i | i \in I\}$, then for each $i \in I$ there exists a prime ideal Q_i of R such that $Q_i \subseteq M_i$, $S_{M_i} = R_{Q_i}$ and R_{M_i}/Q_{M_i} is a valuation domain. Moreover for each $T \in [R, S]$ and for each prime ideal Q of T, set $P = Q \cap R$, then $T_Q = R_P$.

We now present the titular result which is the most important of our paper, as it constitute our main generalization of Gilmer's result [5, Theorem 2.3] who proved, among other things, that if R is an integral domain with quotient field K, then $R \subseteq K$ has FICP (in the terminology of Gilmer, R is an FC-domain) iff $R \subseteq R'$ and $R' \subseteq K$ have FICP.

Theorem 2.4 *Let $R \subseteq S$ be a ring extension such that R^* is semilocal with finite Krull dimension. Then $R \subseteq S$ has FICP if and only if each of the extensions $R \subseteq R^*$ and $R^* \subseteq S$ has FICP.*

The proof of this theorem breaks into two lemmas.

Lemma 2.5 *Let (R, S) be a normal pair. Set $Max(R) = \{M_i | i \in I\}$ and for each $i \in I$, let Q_i the prime ideal of R such that $S_{M_i} = R_{Q_i}$. For a ring $T \in [R, S]$ and a maximal ideal Q of T, there exists $i \in I$ such that $Q \cap R \in [Q_i, M_i]$.*

Proof Let $T \in [R, S]$ and $Q \in \mathrm{Max}(T)$. Set $P = Q \cap R$. There exists a maximal ideal M_i of R such that $P \subseteq M_i$. Let P_i (resp., Q_i) be the prime ideal of R such that $T_{M_i} = R_{P_i}$ (resp., $S_{M_i} = R_{Q_i}$). Such ideals P_i and Q_i exist since (R, T) and (R, S) are normal pairs. As $QT_{M_i} \in \mathrm{Max}(T_{M_i})$, then $QT_{M_i} = P_i R_{P_i}$. Thus $P = P_i$. Now since $R_{M_i} \subseteq T_{M_i} \subseteq S_{M_i}$, then we obtain $P_i = P \in [Q_i, M_i]$, the desired conclusion. □

Lemma 2.6 *Let $R \subseteq S$ be a ring extension. Suppose that $R^* \subseteq S$ has FICP. Then*:

(i) *(R, S) is a residually algebraic pair.*
(ii) *For each $T \in [R, S]$, let $J = T \cap R^*$. Then (J, T) is a normal pair.*

Proof (i) To prove that (R, S) is a residually algebraic pair, it is enough to show that (R^*, S) is a residually algebraic pair [2, Remark 2.2]. For this end, let $T \in [R^*, S]$ and $Q \in \mathrm{Spec}(T)$. Set $P = Q \cap R^*$. Since $R^* \subseteq S$ has FICP then so is $R^* \subseteq T$. According to Proposition 2.1(ii) $R^*/P \subseteq T/Q$ has also FICP. Assume by way of contradiction that $R^*/P \subseteq T/Q$ is not algebraic. Then there exists an element t of T/Q transcendental over R^*/P. Hence, we obtain $(R^*/P)[t] \supset (R^*/P)[t^2] \supset \ldots$ a strictly descending chain of intermediate rings between R^*/P and T/Q, a contradiction.

(ii) Since $[J, T] \subseteq [R, S]$ and (R, S) is a residually algebraic pair (by assertion (i)), then $[J, T]$ inherits the "residually algebraic pair" property from $[R, S]$. Now our task is to show that J is integrally closed in T. Indeed, let $x \in T$ such that x is integral over J, then x is integral over R and hence $x \in T \cap R^* = J$. The desired conclusion. □

Proof of Theorem 2.4 It is clear that the stated condition is necessary. For sufficiency, assume that $R \subseteq R^*$ and $R^* \subseteq S$ have FICP. Set $\mathrm{Max}(R^*) = \{M_k, 1 \leq k \leq n\}$. For each $1 \leq k \leq n$ there exists a prime ideal Q_k of R^* such that $Q_k \subseteq M_k$ and $S_{M_k} = {R^*}_{Q_k}$. Now, let $T_0 = R \subseteq T_1 \subseteq \cdots \subseteq S$ be a chain of intermediate rings in $[R, S]$. Set $\mathrm{Max}(T_i) = \{m_{ij}, j \in I_i\}$, $J_i = T_i \cap R^*$ for each i and let $p_{ij} = m_{ij} \cap J_i$, $j \in I_i$. By Lemma 2.6(ii), we have shown that (J_i, T_i) is a normal pair for each i, so that $(T_i)_{m_{ij}} = (J_i)_{p_{ij}}$ and $T_i = \cap_{j \in I_i} (J_i)_{p_{ij}}$. Because $R \subseteq R^*$ has FICP, the considered chain has the following finite contraction, say: $J_0 = R \subseteq J_1 \subseteq \cdots \subseteq J_s = R^*$. Hence, to show that $R \subseteq S$ has FICP it suffices to show that the set $\mathcal{F} = \{p_{ij} | j \in I_i, 1 \leq i \leq s\}$ is finite. Denote by T_i^* the integral closure of T_i in S. Of course $T_i \subseteq T_i^*$ is an integral extension, so there exists a maximal ideal ${m_{ij}}^*$ in T_i^* lying over m_{ij}. Set ${p_{ij}}^* = {m_{ij}}^* \cap R^*$, notice that (R^*, S) is a normal pair (Lemma 2.6), then by Lemma 2.5, ${p_{ij}}^* \in [Q_k, M_k]$ for some $1 \leq k \leq n$. On the other hand, $R^*_{M_k}/(Q_k)_{M_k}$ is a valuation domain with finite Krull dimension, then $[Q_k, M_k]$ is a totally ordered set which is finite for each $1 \leq k \leq n$. Finally, notice that $p_{ij} = {p_{ij}}^* \cap J_i$. Thus, $\mathcal{F} \subseteq \cup_{1 \leq i \leq s} \cup_{1 \leq k \leq n} [Q_k \cap J_i, M_k \cap J_i]$. Therefore, $\mathcal{F}$ is finite, and this completes the proof. □

In [5], Gilmer defined the term "FC-domain" which means an integral domain R such that $[R, qf(R)]$ has FICP.

As a consequence of Theorem 2.4, we recover Gilmer's result.

Corollary 2.7 [5, Theorem 3.1] *Let R be an integral domain with quotient field K. Then R is an FC-domain if and only if each of the extensions $R \subseteq R'$ and $R' \subseteq K$ has FICP.*

In view of Theorem 2.4, one asks whether "FICP" is transitive. The next example provides extensions of domains $R \subseteq T \subseteq S$ such that both of $R \subseteq T$ and $T \subseteq S$ have FICP, while $R \subseteq S$ does not have PICP.

Example 2.8 Let $R = \mathbb{Z}_{(2)}$, $T = qf(R) = \mathbb{Q}$ and $S = \mathbb{Q}(i)$, where i is the complex number such that $i^2 = -1$. The extensions $R \subseteq T$ and $T \subseteq S$ have FICP since R is a valuation domain with finite spectrum and $[S : T] < +\infty$. On the other hand, suppose that $R \subseteq S$ has FICP and consider $T_k = \mathbb{Z}_{(2)} + 2^k i\mathbb{Z}_{(2)}$ for each positive integer k. It is clear that $T_k \subseteq T_{k+1}$ and $T_k \in [R, S]$ for each k. Hence, there exist positive integers $k \neq l$ such that $T_k = T_l$. In particular, there exist $a, b \in \mathbb{Z}_{(2)}$ such that $2^k i = a + b2^l i$, a contradiction.

We close this section by an application of our results to a well known class of domains, namely pullbacks. Let T be an integral domain, I an ideal of T and D a subring of T/I. Consider the *pullback construction*:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I \end{array}$$

Following [3], we say that R is the domain of the (T, I, D) construction, and we set $R := (T, I, D)$.

Notice that there is a one-to-one correspondence between $[R, T]$ and $[R/I, T/I]$. Then $R \subseteq T$ has FICP iff $D \subseteq T/I$ has FICP. Thus we have:

Corollary 2.9 *Let $R := (T, M, D)$ such that M is a maximal ideal of T and D is a domain with quotient field $K = T/M$. Then the extension $R \subseteq T$ has FICP iff D' is a Prüfer domain with finite spectrum and $D \subseteq D'$ has FICP.*

Proof Follows from Corollary 2.7 and [5, Theorem 1.5]. □

We next state a meaningful upshot of Lemma 2.6.

Corollary 2.10 *Let $R := (T, M, k)$ such that M is a maximal ideal of T and k is a subfield of the field $K = T/M$. Then $R \subseteq T$ has FICP iff $[K : k] < +\infty$.*

Proof It remains to prove that $k \subseteq K$ has FICP iff $k \subseteq K$ is a finite field extension. The "only if" half is trivial. For the "if" half it follows readily from Lemma 2.6(i) that $k \subseteq K$ is an algebraic extension. Hence, it will suffice to show that K is a finite k-module. By way of contradiction, we seek $t_1 \in K \setminus k$ such that $k[t_1] \neq K$. Inductively we can find $(t_n)_{n \geq 2}$ a sequence of elements of K such that $t_{n+1} \in K \setminus k[t_1, \ldots, t_n]$ for each integer $n \geq 2$. This gives rise to the following strictly ascending chain of rings between k and K

$$k \subset k[t_1] \subset \cdots \subset k[t_1, \ldots, t_n] \subset \cdots \subset K,$$

contradicting that $k \subseteq K$ has FICP. □

Despite the motivating result [5], we have seen that $R \subseteq qf(R)$ has FICP iff $R \subseteq R'$ and $R' \subseteq qf(R)$ satisfy FICP iff $R \subseteq R'$ has FICP and R' is a Prüfer domain with finite spectrum. In what follows we provide an example of an extension $R \subseteq S$ such that $R \subseteq S$ has FICP but R^* is not a Prüfer domain. This example depends ultimately on the pullback techniques.

Example 2.11 Let $k \subseteq K$ be an extension of fields such that $[K : k] < +\infty$ and let X, Y two indeterminates over k. Set $m = (X, Y)K[X, Y]$, $n = (X + 1)K[X, Y]$, $U = K[X, Y]\backslash\{m \cup n\}$ and $T = U^{-1}K[X, Y]$. Then T is not a Prüfer domain. Moreover, T is semi local with two maximal ideals $M = U^{-1}m$ and $N = U^{-1}n$ and dim $(T) = 2$. Now, let $R = (T, M, k)$ and $S = T_M$. One check easily that $[T, S] = \{T, S\}$. Then $T \subseteq S$ has FICP. On the other hand, since $[K : k] < +\infty$, then by Corollary 2.10 $R \subseteq T$ has FICP. Finally, we prove that $R \subseteq S$ has FICP. By Theorem 2.4, it suffices to show that $R^* = T$. It is clear that $R^* \subseteq T' = T$ since T is integrally closed. Also, we have $k \subseteq K$ is an algebraic extension since $[K : k] < +\infty$. An application of [3, Lemme 2], shows that $R \subseteq T$ is an integral extension which implies $T \subseteq R^*$. Thus $R^* = T$, as we wished to show.

References

1. Anderson, D.D., Dobbs, D.E., Mullins, B.: The primitive element theorem for commutative algebras. Houston. J. Math. **25**, 603–623 (1999)
2. Ayache, A., Jaballah, A.: Residually algebraic pairs of rings. Math. Z. **225**, 49–65 (1995)
3. Cahen, P.-J.: Couples d'anneaux partageant un idéal. Arch. Math. **51**, 505–514 (1988)
4. Davis, E.D.: Overrings of commutative rings III: Normal pairs. Trans. A. M. S **182**, 1175–1185 (1973)
5. Gilmer, R.: Some finiteness conditions on the set of overrings of an integral domain. Proc. Am. Math. Soc. **131**(8), 2337–2346 (2003)
6. Jaballah, A.: Finiteness of the set of intermediary rings in normal pairs. Saitama Math. J. **17**, 59–61 (1999)
7. Kaplansky, I.: Commutative Rings, 2nd edn. University of Chicago Press, Chicago (1974)